Physics Letters B 692 (2010) 143–151

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Compact QED tree-level amplitudes from dressed BCFW recursion relations

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ARTICLE INFO

Article history:
Received 1 June 2010
Accepted 8 July 2010
Available online 16 July 2010
Editor: A. Ringwald

Keywords:
Gauge theory
QED
Amplitudes in field theory

ABSTRACT

We construct a modified on-shell BCFW recursion relation to derive compact analytic representations of tree-level amplitudes in QED. As an application, we study the amplitudes of a fermion pair coupling to an arbitrary number of photons and give compact formulae for the NMHV and N^2MHV case. We demonstrate that the new recursion relation reduces the growth in complexity with additional photons to be exponential rather than factorial.

1. Introduction

Recent years have seen great progress in our understanding of the underlying structures in gauge theories. On-shell amplitudes are observed to have a much simpler form than their Feynman diagram representations would suggest. Understanding the origin of these structures allows us to construct alternative methods which reproduce the simplicity without the need for large intermediate expressions. On-shell methods like unitarity [1,2] and BCFW recursion [3,4] allow us to study multi-particle and multi-loop amplitudes in a wide range of theories, particularly in theories with a high degree of super-symmetry. A well-studied example of this is $\mathcal{N}=4$ super Yang–Mills (SYM) where a rich structure of symmetries has been uncovered in the planar limit [5,6].

Studies of unordered theories, such as gravity, require us to look beyond the planar limit. Investigations into the UV properties of perturbative amplitudes $\mathcal{N}=8$ supergravity and its connection with $\mathcal{N}=4$ SYM [7,8] have often uncovered new tree-level structures. For recent examples see [9–13]. Analyses of these tree level amplitudes have demonstrated that additional simplifications occur after one obtains expressions from the ordered case via permutation sums. An example of such simplifications is an improved behaviour of the $\mathcal{N}=8$ supergravity tree level amplitudes over that of $\mathcal{N}=4$ SYM amplitudes under large complex deformations of the BCFW shift [14]. For explicit expressions of $\mathcal{N}=8$ supergravity amplitudes see [15,16] ([16] was obtained by solving the BCFW recursion relations in a way similar to $\mathcal{N}=4$ SYM [17].) The latter can be used, through unitarity, to demonstrate the vanishing of triangle coefficients at one loop [18,19]. A result that has also been obtained using a string based approach [20]. An interesting spin-off of this no-triangle property in $\mathcal{N}=8$ supergravity was the observation that similar cancellations persist in another gauge theory, this time without super-symmetry, QED [21].

It is also interesting that in such theories the standard on-shell BCFW recursion does not yield the most compact representation of the amplitude. In this Letter we describe how to use the new information characterising additional gauge cancellations to construct a modified recursion relation with fewer number of terms.

Our approach involves a modification to the BCFW recursion relation by changing the form of the integration kernel of the contour integral. In the context of gravity amplitudes Spradlin, Volovich and Wen constructed a recursive system which led to compact expression [22]. This system can be interpreted as adding a single propagator term into the Cauchy integral. Similar modifications have also been considered recently in the context of boundary terms in BCFW recursion [23] and a re-examination of the U(1) decoupling relation [24]. In this Letter we extend these ideas to the case where multiple propagator factors can be used to modify the recursion relation without introducing a boundary term.

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0370-2693
doi:10.1016/j.physletb.2010.07.021

The upshot is that where applicable, the method we propose yields more compact results compared to those computed via the standard BCFW recursion, and in addition the large z behaviour of the amplitudes becomes manifest.

Our Letter is organised as follows. Firstly we review the Feynman representation of the tree-level QED amplitudes we will study and their improved scaling behaviour under the BCFW complex momentum shift. We then describe the construction of the dressed recursion relation that absorbs certain BCFW diagrams into a modified propagator. In Section 3.1 we re-derive the compact MHV formula of Kleiss and Stirling [25] using the modified recursion. We then derive new compact formulae for the NMHV and N^{2}MHV amplitudes and study their improved combinatoric behaviour. In Section 3.5 we demonstrate the method applies equally to amplitudes with a massive scalar before we reach our conclusions.

2. Tree-level QED amplitudes

In this section we review the tree-level amplitudes of a fermion or massive scalar pair coupling to an arbitrary number of photons and summarise their behaviour under BCFW shifts.

2.1. The $q\bar{q}+n(\gamma)$ amplitudes

The main object we will study in this Letter are the well-known tree-level amplitudes with a fermion pair coupling to an arbitrary number of photons,

$$q^-(k_q)+\bar{q}^+(k_{\bar{q}})+\gamma^{h_1}(k_1)+\cdots+\gamma^{h_n}(k_n)\to 0, \tag{1}$$

where h_i is the helicity of ith photon. The remaining amplitudes with opposite helicity fermions can be obtained via parity symmetry.

These amplitudes were originally computed in [25] and are given by

$$A^{\text{tree}}_{n;q}\big(q^-,\bar{q}^+;1^{h_1},\dots,n^{h_n}\big)=\frac{i}{\prod_{j=1}^n\langle\xi_j k_j\rangle}\sum_{\sigma\in\mathfrak{S}_n}F_{n;q}\big(q^-,\bar{q}^+;\sigma(1)^{h_{\sigma(1)}},\dots,\sigma(n)^{h_{\sigma(n)}}\big), \tag{2}$$

$$F_{n;q}\big(q^-,\bar{q}^+;1^{h_1},\dots,n^{h_n}\big)=\langle a_1 q\rangle[\bar{q}b_n]\prod_{i=1}^{n-1}\frac{\langle a_i|q+K_{1,i}|b_{i+1}]}{(q+K_{1,i})^2}, \tag{3}$$

where $\{\xi_k\}$ is a set of light-like reference vectors and $K_{1,i}=\sum_{j=1}^i k_i$. We have also defined

$$a_i=\frac{1+h_i}{2}\xi_i+\frac{1-h_i}{2}k_i,\qquad b_i=\frac{1+h_i}{2}k_i+\frac{1-h_i}{2}\xi_i. \tag{4}$$

The definitions of for spinor products follow the standard conventions used in the QCD literature and are summarised in Appendix A.

In the MHV case the amplitude one can show [26] that the amplitude takes the following simplified form,

$$A^{\text{tree}}_{n;q}\big(q^-,\bar{q}^+;1^-,2^+,\dots,n^+\big)=i\frac{\langle q\bar{q}\rangle^{n-2}\langle 1q\rangle^2}{\prod_{\alpha=2}^n\langle q\alpha\rangle\langle\bar{q}\alpha\rangle}. \tag{5}$$

2.2. The $S\bar{S}+n(\gamma)$ amplitudes

The tree-level amplitudes with a massive (complex) scalar coupling to an arbitrary number of photons are given as [21]

$$A^{\text{tree}}_{n;S}(S,\bar{S};k_1,\dots,k_n)=i\sum_{\sigma\in\mathfrak{S}_n}F_{n;S}(S,\bar{S};k_{\sigma(1)},\dots,k_{\sigma(n)}), \tag{6}$$

of an amplitude defined from the partition of the n ordered external legs partitioned in group of at most length two

$$F_{n;S}(S,\bar{S};k_{\sigma(1)},\dots,k_{\sigma(n)})=\sum_{\substack{a_1+\cdots+a_r=n\\ a_k\in\{1,2\}}}\prod_{s=1}^{r}\frac{\epsilon_{\sigma(a_1+\cdots+a_{s-1}+1)}\cdot H(a_s)}{(p_a+\sum_{j=1}^{a_1+\cdots+a_s}k_{\sigma(j)})^2-\mu^2}, \tag{7}$$

with

$$H(a_s)=\begin{cases} q+\sum_{j=1}^{a_1+\cdots+a_{s-1}}k_{\sigma(j)} & \text{if } a_s=1,\\ \epsilon_{\sigma(a_1+\cdots+a_s)} & \text{if } a_s=2.\end{cases} \tag{8}$$

Because of the cubic and quartic vertices, this amplitude is a much larger sum of terms than the fermionic case. The number can be expressed as a sum over $n!\times F_{n+1}$ where F_r is the Fibonacci number of order r (such that $F_0=F_1=1$ and $F_2=2$).

2.3. Large z scaling

We consider the polynomial behaviour of tree level amplitudes listed above under large values of the complex parameter used in the BCFW recursion relations. We define such a complex momentum shift as $\langle a,b]$ where[1]

$$\hat{a}^\mu=a^\mu-\frac{z}{2}\langle a|\gamma^\mu|b],\qquad \hat{b}^\mu=b^\mu+\frac{z}{2}\langle a|\gamma^\mu|b]. \tag{9}$$

[1] Note that this definition differs from other instances in the literature where $\langle a,b]$ corresponds to a shift vector of $\langle b|\gamma^\mu|a]$.

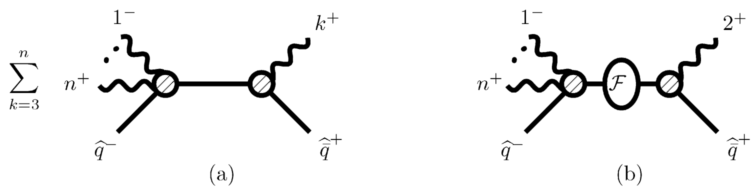

Fig. 1. (a) The $(n-1)$ diagrams contributing to the standard BCFW recursion relation for the $q\bar{q}+n$-photon amplitudes. (b) The modified recursion containing only a single term.

Recently it was shown that one-loop multi-photon amplitudes have a surprisingly simple structure [21]. This can be explained through analysing the large z behaviour of the tree level amplitudes entering generalised cuts in the loop amplitude, a technique that has successfully uncovered similar cancellations in gravity theories [18]. The key insight of [21] was to demonstrate an improvement in the behaviour of the $q\bar{q}+$ photon tree amplitudes under large values of z when shifting the fermion pair,

$$A_{n;q}^{\text{tree}}\left(q^-,\bar{q}^+;1^{h_1},\ldots,n^{h_n}\right) \overset{\langle q,\bar{q}];z\to\infty}{\sim} \frac{C_\infty}{z^{n-1}}. \tag{10}$$

This improved scaling only appears after the permutation sum has been performed and is independent of the helicities of the photon lines.

It was also observed that the amplitudes with a pair of massive scalars also share the same property,

$$A_{n;S}^{\text{tree}}\left(S^-,\bar{S}^+;1^{h_1},\ldots,n^{h_n}\right) \overset{\langle S^\flat,\bar{S}^\flat];z\to\infty}{\sim} \frac{C_\infty}{z^{n-2}}. \tag{11}$$

3. Dressing the BCFW relation

The on-shell BCFW recursion relation can be derived by considering a complex contour integral over the function $A(z)/z$:

$$A_\infty = \frac{1}{2\pi i}\oint dz\frac{A(z)}{z} = A(0) - \sum_{\text{residues } z_k} A_L(z_k)\frac{i}{P_k^2}A_R(z_k), \tag{12}$$

where $z_k = P_k^2/\langle a|P_k|b]$ for a $\langle a,b]$ shift and the momentum P_k flows from right to left.[2] the term A_∞ is zero as long as $A(z) \overset{z\to\infty}{\to} \mathcal{O}(1/z)$ or better. For the QED amplitudes considered above we observed that under certain shifts the large z behaviour was much better than this minimum requirement. This allows us the freedom to consider a new integral which will still evaluate to zero:

$$0 = \frac{1}{2\pi i}\oint dz\frac{\alpha - z}{\alpha}\frac{A(z)}{z} = A(0) - \sum_{\text{residues } z_k} A_L(z_k)\frac{\alpha - z_k}{\alpha}\frac{i}{P_k^2}A_R(z_k). \tag{13}$$

Since we are free to choose α we can use this factor to cancel one of the poles in $A(z)$ and therefore reduce the number of terms in $A(0)$ compared to the representation of Eq. (12). The fact that an improved large z behaviour of the amplitudes can be used to derive simplified expressions for tree-level amplitudes has been observed previously in [14]. For one inserted dressing factor our formula is essentially identical to the approach used in [22] to derive relations between supergravity amplitudes. For our QED amplitudes under the $\langle q,\bar{q}]$ shift we have the modified boundary behaviour of z^{1-n} and so we can reduce the number of diagrams in the recursion relation by introducing $(n-2)$ additional propagator factors. The recursion relation then takes the following form:

$$0 = -\frac{1}{2\pi i}\oint\frac{dz}{z}A(z)\prod_{l=1}^{n-2}\frac{z_l - z}{z_l} = A(0) - \sum_{\text{residues } z_k} A_L(z_k)\frac{i\mathcal{F}_n(P_k)}{P_k^2}A_R(z_k), \tag{14}$$

where

$$\mathcal{F}_n(P_k) = \prod_{l=1}^{n-2}\frac{z_l - z_k}{z_l} = \frac{1}{\langle q|P_k|\bar{q}]^{n-2}}\prod_{l=1}^{n-2}\frac{\langle q|P_k(P_l - P_k)P_l|\bar{q}]}{P_l^2}. \tag{15}$$

We see that the dressing factors contain all the explicitly removed propagators, P_l^2, thus ensuring that the amplitude has the correct pole structure. It is interesting to note that under a subsequent $\langle q,\bar{q}]$ shift (15) falls off as z^{2-n} due to these additional propagator factors. Hence if we compare the expression for an amplitude computed from the dressed BCFW relation (14) to that computed from the standard one (12), we notice the following: apart from consisting of fewer terms each term in the former expression will have the improved large z behaviour. So if it is known that an amplitude has a certain large z behaviour, this can be made manifest term by term using the dressed recursion relations. In addition, the formula obtained in this way will consist of fewer terms compared to a formula obtained from the standard recursion relations.

[2] The factor of i in the propagator is specific to gluons and scalars. Fermion propagators require and additional factor of $-i$ as we will see later.

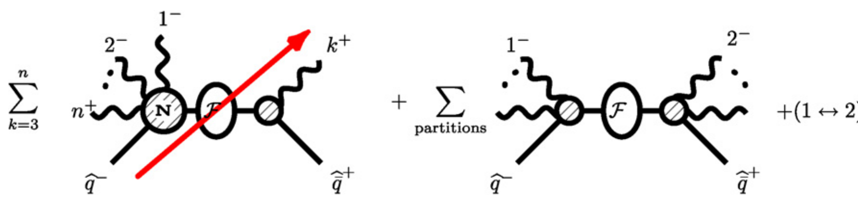

Fig. 2. Of the two topologies contributing to the NMHV photon amplitudes all of the diagrams involving lower point NMHV vertices can be eliminated by dressing the recursion relation.

3.1. Re-derivation of the Kleiss–Stirling MHV amplitude

In this section we re-derive the photon-MHV amplitude of Eq. (5).

We choose the $\langle q, \bar{q}]$ shift and take the momentum of the negative helicity to photon to be p_1. Since the amplitude is unordered this is done without loss of generality. We choose the $(n-2)$ dressing factors such that we cancel contributions from the two-particle channels with the anti-quark and the positive helicity photons $p_3, \ldots, p_n$. In the MHV case we must have at least one negative helicity particle in each tree level sub-amplitude or we will get a vanishing contribution. The only exception to this is the three-point $\overline{\text{MHV}}$amplitude. There are therefore $(n-1)$ contributing terms to the standard BCFW recursion relation for this helicity configuration as shown in Fig. 1. However the modified recursion relation has only a single contribution from the $P_{\bar{q}2}$ channel (we denote $P^{\mu}_{\bar{q}2} = \bar{q}^{\mu} + k^{\mu}_2$ and $(P_{\bar{q}2})^2 = s_{\bar{q}2}$) where the dressing factor is given by:

$$\mathcal{F}_n(P_{\bar{q}2}) = \prod_{k=3}^{n} \frac{\langle q|P_{\bar{q}2}(P_{\bar{q}k} - P_{\bar{q}2})P_{\bar{q}k}|\bar{q}]}{\langle q|P_{\bar{q}2}|\bar{q}]s_{\bar{q}k}} = \frac{\langle q\bar{q}\rangle^{n-2}}{\langle q2\rangle^{n-2}} \prod_{k=3}^{n} \frac{\langle 2k\rangle}{\langle \bar{q}k\rangle}. \tag{16}$$

The modified recursion relation then simply becomes:

$$A^{\text{tree}}_{n;q}(q^-, \bar{q}^+; 1^-, 2^+, \ldots, n^+) = -iA^{\text{tree}}_{n-1;q}(\hat{q}^-, \hat{P}^+_{\bar{q}2}; 1^-, 3^+ \ldots, n^+) \frac{i\mathcal{F}_n(P_{\bar{q}2})}{s_{\bar{q}2}} A^{\text{tree}}_{1;q}(-\hat{P}^-_{\bar{q}2}, \hat{\bar{q}}^+; 2^+). \tag{17}$$

This can be solved inductively using the Kleiss–Stirling formula (5) as an ansatz. In order to prove (5) recursively we need the following two three-point amplitudes,

$$A^{\text{tree}}_{1;q}(q^-, \bar{q}^+; 1^-) = i\frac{\langle 1q\rangle^2}{\langle q\bar{q}\rangle}, \tag{18}$$

and

$$\tilde{A}^{\text{tree}}_{1;q}(q^-, \bar{q}^+; 1^+) = i\frac{[1\bar{q}]^2}{[q\bar{q}]}, \tag{19}$$

which follow from the vertices in the QED Lagrangian. Notice that the first formula agrees with (5) for $n = 1$. It is then sufficient to assume that (5) holds for $(n-1)$ photons and show that it holds for n photons. Indeed, using

$$A^{\text{tree}}_{n-1;q}(\hat{q}^-, \hat{P}^+_{\bar{q}2}; 1^-, 3^+, \ldots, n^+) = i\frac{\langle q2\rangle^{n-3}\langle 1q\rangle^2[\hat{P}_{\bar{q}2}\bar{q}]}{[2\bar{q}]\prod_{k=3}^{n}\langle qk\rangle\langle 2k\rangle}, \tag{20}$$

and (see (A.5))

$$A^{\text{tree}}_{1;q}(-\hat{P}^-_{\bar{q}2}, \hat{\bar{q}}^+; 2^+) = \frac{[2\bar{q}]^2}{[\hat{P}_{\bar{q}2}\bar{q}]}, \tag{21}$$

and (16) and plugging them into (17) we immediately obtain

$$A^{\text{tree}}_{n;q}(q^-, \bar{q}^+; 1^-, 2^+, \ldots, n^+) = i\frac{\langle q1\rangle^2}{\langle q\bar{q}\rangle} \prod_{k=2}^{n} \frac{\langle q\bar{q}\rangle}{\langle qk\rangle\langle \bar{q}k\rangle}. \tag{22}$$

This shows us that the modified recursion relations give an efficient way to derive compact expressions for the QED amplitudes. The simplifications from the permutation sums are essentially factored into the $\mathcal{F}$ functions modifying the propagator.

3.2. NMHV amplitudes

We now turn our attention to the NMHV amplitudes with two negative helicity photons. The first non-trivial of these occurs for $n = 4$ and a compact form for it has previously been derived using a standard BCFW shift of the anti-fermion and a negative helicity photon, $\langle 1, \bar{q}]$, [26][3]:

[3] In order not to make the notation too heavy, in the following we will abbreviate e.g. p_j by j. We hope that this does not lead to confusion.

$$A^{\text{tree}}_{4;q}\big(q^-,\bar{q}^+;1^-,2^-,3^+,4^+\big)$$
$$= P\big(1^-,2^-,3^+,4^+\big)+P\big(1^-,2^-,4^+,3^+\big)+Q\big(1^-,2^-,3^+,4^+\big)+Q\big(1^-,2^-,4^+,3^+\big)+R\big(1^-,2^-,4^+,3^+\big), \tag{23}$$

where

$$P\big(q^-,\bar{q}^+;1^-,2^-,3^+,4^+\big)=\frac{\langle 1|\bar{q}+3|q]\langle 1|\bar{q}+3|4]^2}{s_{\bar{q}13}[q2]\langle\bar{q}3\rangle\langle 3|\bar{q}+1|q]\langle 1|\bar{q}+p3|2]}, \tag{24}$$

$$Q\big(q^-,\bar{q}^+;1^-,2^-,3^+,4^+\big)=\frac{\langle q1\rangle^2[3\bar{q}]^2\langle 1|3+2|\bar{q}]}{s_{\bar{q}23}\langle q4\rangle[1\bar{q}]\langle 4|3+2|\bar{q}]\langle 1|\bar{q}+3|2]}, \tag{25}$$

$$R\big(q^-,\bar{q}^+;1^-,2^-,3^+,4^+\big)=\frac{s_{234}[q\bar{q}]^2\langle 2|q+1|\bar{q}]^2}{\langle 3|q+1|\bar{q}]\langle 4|q+1|\bar{q}]\langle 3|\bar{q}+1|q]\langle 4|\bar{q}+1|q][1\bar{q}][q1]}. \tag{26}$$

An observation one can make is that the functions P and Q both scale as $1/z^2$ rather than the $1/z^3$ behaviour of the full amplitude in the large z limit. Using the dressed recursion relation and a $\langle q,\bar{q}]$ shift we can derive an alternative formula in which each term scales as $1/z^3$.

We begin by choosing the $(n-2)$ diagrams with a three-point $\overline{\text{MHV}}$-amplitude to vanish by using the same dressing factor as in the MHV case,

$$\mathcal{F}_n(P)=\prod_{l=3}^{n}\frac{z_l-z_P}{z_l}=\frac{1}{\langle q|P|\bar{q}]^{n-2}}\prod_{l=3}^{n}\frac{\langle q|P(\bar{q}-P)|l\rangle}{\langle\bar{q}l\rangle}. \tag{27}$$

The recursion relation then takes the form of a sum over products of two MHV amplitudes:

$$A^{\text{tree}}_{n;q}\big(q^-,\bar{q}^+,1^-,2^-,3^+,\ldots,n^+\big)$$
$$=\sum_{\sigma\in S_2}\sum_{P_1|P_2}-iA^{\text{tree}}_{n_1+1;q}\big(\hat{q}^-,\hat{Q}_1^+,1^-,\{P_1\}\big)\frac{i\mathcal{F}_n(Q_1)}{Q_1^2}A^{\text{tree}}_{n_2+1;q}\big(-\hat{Q}_1^-,\hat{\bar{q}}^+,2^-,\{P_2\}\big)$$
$$\equiv\sum_{\sigma\in S_2}\sum_{P_1|P_2}F_{n_1,n_2}\big(q^-,\bar{q}^+,1^-,2^-,\{P_1\},\{P_2\}\big). \tag{28}$$

There is only a single topology left for the full NMHV amplitude as shown in Fig. 2:

$$F_{n_1,n_2}\big(q^-,\bar{q}^+,1^-,2^-,\{P_1\},\{P_2\}\big)=A^{\text{tree}}_{n_1+1;q}\big(\hat{q}^-,\hat{Q}_1^+,1^-,\{P_1\}\big)\frac{\mathcal{F}_n(Q_1)}{Q_1^2}A^{\text{tree}}_{n_2+1;q}\big(-\hat{Q}_1^-,\hat{\bar{q}}^+,2^-,\{P_2\}\big), \tag{29}$$

where

$$Q_1=\bar{q}+2+\sum_{k\in P_2}k. \tag{30}$$

We also define n_1 and n_2 to be the number of positive helicity photons in the left and right amplitudes. Using the expressions for the MHV amplitudes and expanding the shifted spinors we find

$$F_{n_1,n_2}\big(q^-,\bar{q}^+,1^-,2^-,\{P_1\},\{P_2\}\big)=\frac{-i\langle q1\rangle^2\langle 2|Q_1|\bar{q}]^2}{Q_1^2(Q_1-\bar{q})^2\langle q|Q_1|\bar{q}]}\prod_{k\in P_1}\frac{\langle q|Q_1(Q_1-\bar{q})|k\rangle}{\langle qk\rangle\langle k\bar{q}\rangle\langle k|Q_1|\bar{q}]}\prod_{k\in P_2}\frac{(Q_1-\bar{q})^2}{\langle\bar{q}k\rangle\langle k|Q_1|\bar{q}]}. \tag{31}$$

Restricting ourselves to the $n=4$ NMHV amplitude we can explicitly write

$$A^{\text{tree}}_{4;q}\big(q^-,\bar{q}^+;1^-,2^-,3^+,4^+\big)$$
$$=F_4^1\big(1^-,2^-,3^+,4^+\big)+F_4^1\big(2^-,1^-,3^+,4^+\big)$$
$$+F_4^2\big(1^-,2^-,3^+,4^+\big)+F_4^2\big(1^-,2^-,4^+,3^+\big)+F_4^2\big(2^-,1^-,3^+,4^+\big)+F_4^2\big(2^-,1^-,4^+,3^+\big), \tag{32}$$

where

$$F_4^1\big(1^-,2^-,3^+,4^+\big)=\frac{is_{234}\langle 2|3+4|\bar{q}]^2}{\langle 3\bar{q}\rangle\langle 4\bar{q}\rangle\langle 3|2+4|\bar{q}]\langle 4|2+3|\bar{q}][1q][1\bar{q}]}, \tag{33}$$

$$F_4^2\big(1^-,2^-,3^+,4^+\big)=\frac{i\langle q1\rangle^2\langle 24\rangle\langle q|1+3|2+4|3\rangle[\bar{q}4]^2}{s_{q13}\langle 3q\rangle\langle 3\bar{q}\rangle\langle 4\bar{q}\rangle\langle q|2+4|\bar{q}]\langle 3|2+4|\bar{q}][\bar{q}2]}, \tag{34}$$

and where it is simple to check that Eqs. (32) and (23) agree numerically.

Comparing these two equations one notices that our new improved recursion relation yields a representation which is actually one term longer than the standard BCFW $\langle\gamma,\bar{q}]$ shift. It turns out that this is not a general feature and it is the purpose of the next section to show that in general (31) is considerably more compact compared to the previously known results.

Table 1
Number of terms obtained from conventional vs dressed BCFW recursion. m is the number of plus helicity photons.

m	1	2	3	4	5	6	7	8	9	10
$f_m^{\text{NMHV}}/2$	1	5	22	103	546	3339	23500	188255	1694806	16949083
$f_m^{\text{NMHV,dressed}}$	2	6	14	30	62	126	254	510	1022	2046

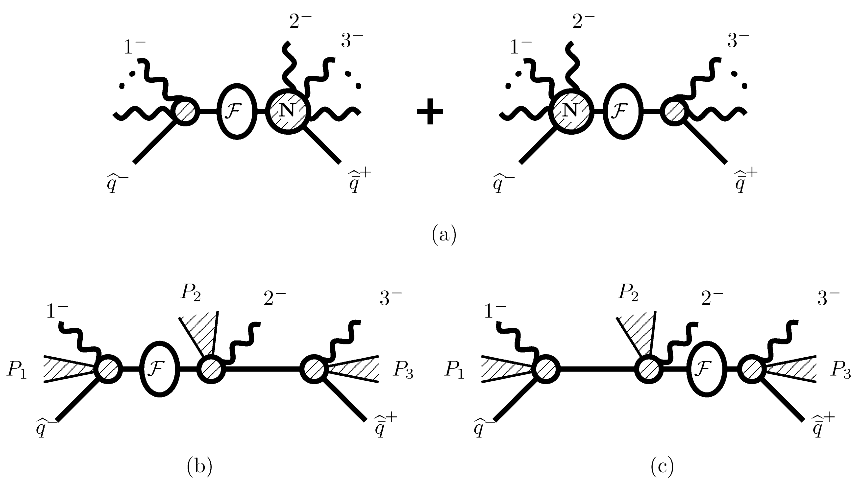

Fig. 3. (a) N^2MHV amplitude as the sum over products of MHV and NMHV amplitudes in the dressed recursion relation. (b) Diagrammatic representation of the function G. (c) Diagrammatic representation of the function G'.

3.3. Number of terms in the amplitudes

Let us compare the number of terms for the NMHV amplitudes considered above, and compare between the standard and dressed recursion relation.

Recall that n is the total number of photons. We denote the number of plus helicity photons by m. For the NMHV amplitude, $m = n - 2$. For a BCFW shift which involves the fermion/anti-fermion pair we get the following recurrence relation for the number of terms f_m^{NMHV} in the NMHV amplitude,

$$f_m^{\text{NMHV}} = m f_{m-1}^{\text{NMHV}} + 2g_m, \tag{35}$$

with $g_m = \sum_{i=1}^{m} m!/(m-i)!/i! = 2^m - 1$ and $f_0^{\text{NMHV}} = 0$. Here the factor m comes from the ways of choosing the plus helicity photon on the $\overline{\text{MHV}}_3$ vertex, and g_m counts the number of diagrams built from two MHV vertices. The factor two accounts for the two possible positions of the negative helicity photons. The solution to this recurrence is $f_m^{\text{NMHV}} = 2m!(\sum_{k=0}^{m} 2^k/k! - \sum_{k=0}^{m} 1/k!)$, so f_m^{NMHV} grows factorially for large m,[4]

$$f_m^{\text{NMHV}} \sim 2\big(e^2 - e\big)m!. \tag{36}$$

Let us now study the improvement induced by the dressed recursion relations discussed in Section 3.2. Recall that thanks to the z^{-m-1} falloff for large z we can introduce m dressing factors in such a way that they eliminate the m diagrams homogeneous in the NMHV amplitude. Hence we will obtain an improved relation for the number of terms in the amplitude,

$$f_m^{\text{NMHV,dressed}} = 2g_m = 2\big(2^m - 1\big) \sim 2^{m+1}, \tag{37}$$

which grows *exponentially* instead of *factorially*. While for small values of m one finds a comparable number of terms the advantage of the dressed recursion relations becomes obvious for bigger values of m. Some sample values are given in the following table for illustration. Some sample values are given in Table 1 for illustration.

3.4. N^2MHV amplitudes

In this section we derive closed form analytic expressions for the N^2MHV amplitudes with three negative helicity photons using the dressed recursion relation. Just as in the NMHV case we can use the dressing factor to remove all homogeneous diagrams so that the NMHV amplitude is simply a product of NMHV and MHV amplitudes as shown in Fig. 3(a). We write the dressed recursion relation as:

[4] It is possible to obtain slightly improved expressions, but which still exhibit a factorial growth. For example, shifting the anti-fermion and one of the negative helicity photons as in [26], one obtains $f_m^{\text{NMHV}} \sim (e^2 - e)m!$. Also, by making a convenient choice of the reference spinors in the Kleiss–Stirling formula (3) one finds $f_m^{\text{NMHV}} = (m-1)!$ [21].

$$\begin{aligned}
&A^{\text{tree}}_{n;q}\big(q^-,\bar{q}^+,1^-,2^-,3^-,4^+,\ldots,n^+\big)\\
&=\sum_{\sigma\in S_3/\mathbb{Z}}\sum_{P_L|P_R}-iA^{\text{tree}}_{n_L;q}\big(\hat{q}^-,\hat{Q}_1^+,\sigma(1)^-,\{P_L\}\big)\frac{i\mathcal{F}_n(Q_1)}{Q_1^2}A^{\text{tree}}_{n_R;q}\big(-\hat{Q}_1^-,\hat{\bar{q}}^+,\sigma(2)^-,\sigma(3)^-,\{P_R\}\big)\\
&\quad+\sum_{\sigma\in S_3/\mathbb{Z}}\sum_{P_L|P_R}A^{\text{tree}}_{n_L;q}\big(\hat{q}^-,\hat{Q}_1^+,\sigma(1)^-,\sigma(2)^-,\{P_L\}\big)\frac{\mathcal{F}_n(Q_1)}{Q_1^2}A^{\text{tree}}_{n_R;q}\big(-\hat{Q}_1^-,\hat{\bar{q}}^+,\sigma(3)^-,\{P_R\}\big)\\
&=\sum_{\sigma\in S_3}\Bigg(\sum_{P_1|P_2|P_3}G_{n_1,n_2,n_3}\big(q^-,\bar{q}^+,\sigma(1)^-,\sigma(2)^-,\sigma(3)^-,\{P_1\},\{P_2\},\{P_3\}\big)\\
&\quad+\sum_{P'_1|P'_2|P'_3}G'_{n_1,n_2,n_3}\big(q^-,\bar{q}^+,\sigma(1)^-,\sigma(2)^-,\sigma(3)^-,\{P'_1\},\{P'_2\},\{P'_3\}\big)\Bigg).
\end{aligned}\tag{38}$$

Each G, G' function represents a single term in the final result expressed as a product of an MHV amplitude with the NMHV F function defined in Eq. (31),

$$G_{n_1,n_2,n_3}\big(q^-,\bar{q}^+,1^-,2^-,3^-,\{P_1\},\{P_2\},\{P_3\}\big)=A^{\text{tree}}_{n_1+1;q}\big(\hat{q}^-,\hat{Q}_1^+;1^-,\{P_1\}\big)\frac{\mathcal{F}_n(Q_1)}{Q_1^2}F_{n_2,n_3}\big(-\hat{Q}_1^-,\bar{q}^+,2^-,3^-,\{P_2\},\{P_3\}\big),\tag{39}$$

and

$$G'_{n_1,n_2,n_3}\big(q^-,\bar{q}^+,1^-,2^-,3^-,\{P_1\},\{P_2\},\{P_3\}\big)=F_{n_1,n_2}\big(\hat{q}^-,\hat{Q}_2^+,1^-,2^-,\{P_1\},\{P_2\}\big)\frac{\mathcal{F}_n(Q_2)}{Q_2^2}A^{\text{tree}}_{n_3+1;q}\big(-\hat{Q}_2^-,\bar{q}^+;3^-,\{P_3\}\big),\tag{40}$$

where

$$Q_1=\bar{q}+2+3+\sum_{k\in P_2\cup P_3}k,\qquad Q_2=\bar{q}+3+\sum_{k\in P_3}k.\tag{41}$$

These topologies are written graphically in Figs. 3(b) and (c). These functions can be quickly written in closed form:

$$\begin{aligned}
&G_{n_1,n_2,n_3}\big(q^-,\bar{q}^+,1^-,2^-,3^-,\{P_1\},\{P_2\},\{P_3\}\big)\\
&=\frac{i\langle q1\rangle^2\langle 2|Q_1|\bar{q}]^2\langle 3|Q_2|\bar{q}]^2\langle q|Q_1(Q_1-\bar{q})|3\rangle}{Q_1^2(Q_1-Q_2)^2\langle q|Q_1(Q_2-Q_1)Q_2|\bar{q}][\bar{q}|Q_1Q_2|\bar{q}]\langle q|Q_1|\bar{q}]\langle\bar{q}3\rangle}\\
&\quad\times\prod_{k\in P_1}\frac{\langle q|Q_1(Q_1-\bar{q})|k\rangle}{\langle qk\rangle\langle\bar{q}k\rangle\langle k|Q_1|\bar{q}]}\prod_{k\in P_2}\frac{\langle k|(Q_2-\bar{q})(Q_1-Q_2)Q_1|\bar{q}]}{\langle k\bar{q}\rangle\langle k|Q_2|\bar{q}]\langle k|Q_1|\bar{q}]}\prod_{k\in P_3}\frac{(Q_1-Q_2)^2}{\langle k\bar{q}\rangle\langle k|Q_2|\bar{q}]\langle k|Q_1|\bar{q}]},
\end{aligned}\tag{42}$$

and

$$\begin{aligned}
&G'_{n_1,n_2,n_3}\big(q^-,\bar{q}^+,1^-,2^-,3^-,\{P_1\},\{P_2\},\{P_3\}\big)\\
&=\frac{i\langle q1\rangle^2\langle 2|(Q_1-Q_2)Q_1|q\rangle^2\langle 3|Q_2|\bar{q}]^2\langle q|Q_2(Q_2-\bar{q})|3\rangle}{Q_2^2(Q_2-Q_1)^2(Q_2-\bar{q})^2\langle q|Q_2Q_1|q\rangle\langle q|Q_2|\bar{q}]\langle\bar{q}3\rangle}\prod_{k\in P_1}\frac{\langle q|Q_1(Q_1-Q_2)|k\rangle\langle q|Q_2(Q_2-\bar{q})|k\rangle}{\langle qk\rangle\langle k|Q_2|\bar{q}]\langle k|(Q_1-Q_2)Q_2|q\rangle\langle\bar{q}k\rangle}\\
&\quad\times\prod_{k\in P_2}\frac{(Q_1-Q_2)^2\langle q|Q_2(Q_2-\bar{q})|k\rangle}{\langle k|Q_2|\bar{q}]\langle k|(Q_1-Q_2)Q_2|q\rangle\langle\bar{q}k\rangle}\prod_{k\in P_3}\frac{(Q_2-\bar{q})^2}{\langle k|Q_2|\bar{q}]\langle k\bar{q}\rangle}.
\end{aligned}\tag{43}$$

The full N^2MHV amplitude for $n=6$ contains 186 terms in the dressed case versus 720 for the Feynman diagram computation. The counting of terms can be done as in Section 3.3. For the N^2MHV amplitude obtained form the dressed recursion we arrive at a total number of terms (m is the number of plus helicity photons, i.e. $m=n-3$ for N^2MHV):

$$f^{\mathrm{N}^2\mathrm{MHV}}(m)=12\times 3^m-18\times 2^m+6.\tag{44}$$

We chose to use the same dressing factor as in the NMHV case even though here the $z=z_{3\bar{q}}$ channel (corresponding to $l=3$ in Eq. (27)) vanishes explicitly. Although one could choose it to be another non-zero diagram the saving in complexity would be modest and at the cost of losing the symmetry of the final answer.

3.5. Massive scalar amplitudes

In this section we consider amplitudes with photons and a pair for massive (complex) scalars. The amplitudes with gluons have been computed using a massive BCFW recursion in reference [27]. We can obtain the two-photon amplitude by summing over permutations of the gluon result:

$$\begin{aligned}
A^{\text{tree}}_{4;S}\big(S^+,1^+,2^+,\bar{S}^-\big)&=A^{\text{tree}}_{4;S}\big(S^+,1_g^+,2_g^+,\bar{S}^-\big)+A^{\text{tree}}_{4;S}\big(S^+,2_g^+,1_g^+,\bar{S}^-\big)\\
&=\frac{im^2[12]}{\langle 12\rangle\langle 1|S|1]}+\frac{im^2[12]}{\langle 12\rangle\langle 2|S|2]}.
\end{aligned}\tag{45}$$

Here we derive this four-point amplitude using the dressed recursion and shifting the two massive particles as described by Schwinn and Weinzierl [28]. For this we first define a basis of two massless vectors from the original massive pair:

$$S^\flat = \frac{\gamma(\gamma S + m^2 \bar{S})}{\gamma^2 - m^4}, \qquad \bar{S}^\flat = \frac{\gamma(\gamma \bar{S} + m^2 S)}{\gamma^2 - m^4}, \quad \gamma = S \cdot \bar{S} + \sqrt{(S \cdot \bar{S})^2 - m^4}. \tag{46}$$

We first compute the all-plus configuration which vanishes in the massless limit. This amplitude actually has a further improved boundary behaviour with respect to the universal scaling and goes as $1/z^n$ in the large z limit. We first define the dressing function which we will use for the n-point function:

$$\mathcal{F}_n(z) = \prod_{l=2}^{n} \frac{z_l - z}{z_l}, \tag{47}$$

where

$$z_l = \frac{\gamma \langle l|\bar{S}|l]}{(\gamma - m^2)\langle S^\flat|l|\bar{S}^\flat]}. \tag{48}$$

The $n = 2$ amplitude can then be represented as:

$$\begin{aligned} A^{\text{tree}}_{4;S}\big(S, \bar{S}; 1^+, 2^+\big) &= A^{\text{tree}}_{3;S}\big(\hat{S}, \hat{P}_{\bar{S}1}; 2^+\big) \frac{i\mathcal{F}_2(z_1)}{\langle 1|\bar{S}|1]} A^{\text{tree}}_{3;S}\big(-\hat{P}_{\bar{S}1}, \hat{\bar{S}}; 1^+\big) \\ &= \frac{im^2 \mathcal{F}_2(z_1)\langle S^\flat \bar{S}^\flat\rangle^2 [\bar{S}^\flat 2][\bar{S}^\flat 1]}{\gamma \langle 1|\bar{S}|1]\langle S^\flat 1\rangle\langle S^\flat 2\rangle}. \end{aligned} \tag{49}$$

Using the momentum conservation for the flatted vectors, $(1 + \frac{m^2}{\gamma})(S^\flat + \bar{S}^\flat) + 1 + 2 = 0$, we find:

$$\mathcal{F}_2(z_1) = -\frac{s_{12}}{\langle 2|\bar{S}|2]}, \quad \text{and} \quad \frac{\langle S^\flat|\bar{S}^\flat|2]}{\langle 1|S^\flat|\bar{S}^\flat]} = \frac{\langle S^\flat 1\rangle[12]}{\langle 12\rangle[2\bar{S}^\flat]}. \tag{50}$$

This allows to eliminate S and $\bar{S}$ from the amplitude leaving,

$$A^{\text{tree}}_{4;S}\big(S, \bar{S}; 1^+, 2^+\big) = \frac{im^2[12]^2}{\langle 1|S|1]\langle 2|S|2]}, \tag{51}$$

which matches the standard result. Higher multiplicity amplitudes for $n = 5, 6$ have been checked numerically against Eq. (7) to verify the validity of the dressed recursion. Since there is no simple choice of dressing factors as in the fermion amplitudes an all multiplicity solution to the recursion is more difficult to obtain and we refrain from further generalisations for the time being.

4. Conclusions

In this Letter we have constructed a dressed version of the BCFW recursion relation which allows the computation of compact analytic formula for tree-level amplitudes in QED. The construction relies on the improved boundary scaling property first observed in [21] and can be applied to any situation where the amplitude falls off as $1/z^2$ at the boundary of the integration contour. The new NMHV and N^2MHV amplitude representations are shown to have an exponential rather than factorial growth in the number of terms compared to the standard on-shell recursion.

Since all formulae have the improved scaling behaviour manifest they are much better suited to find the cancellations in loop amplitudes explicitly. They would be particularly useful in finding closed form expression for the multi-photon amplitudes at one-loop for which the current limit is the eight-point MHV amplitude.

Acknowledgements

We are grateful to J. Drummond for stressing that an improved large z behaviour implies relations between different BCFW terms, which initiated this project. We also thank P. Benincasa for interesting discussions and P. Vanhove for useful comments on the manuscript. S.B. acknowledges support from the Helmholtz Gemeinschaft under contract VH-NG-105.

Appendix A. Conventions

We use the standard QCD conventions for the two-component spinor helicity formalism.

$$(p_1 + p_2)^2 = 2p_1 \cdot p_2 = \langle 12\rangle[21]. \tag{A.1}$$

Here

$$\langle 12\rangle = \lambda_1^\alpha \lambda_{2\alpha}, \qquad [21] = \tilde{\lambda}_{2\dot{\alpha}} \tilde{\lambda}_1^{\dot{\alpha}}. \tag{A.2}$$

The extended spinor product is defined as,

$$\langle q|P_k|\bar{q}] = \lambda_q^\alpha P_{k\alpha\dot{\alpha}} \tilde{\lambda}_{\bar{q}}^{\dot{\alpha}}, \tag{A.3}$$

and

$$P_k^2 = P_k \cdot P_k = \frac{1}{2} P_k^{\alpha\dot{\alpha}} P_{k\alpha\dot{\alpha}}. \tag{A.4}$$

When encountering negative momenta in the spinors appearing in the recursion relations we define:

$$|-p] = i|p], \qquad |-p\rangle = i|p\rangle. \tag{A.5}$$

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